

ON HOMOTOPY CATEGORIES OF GORENSTEIN MODULES: COMPACT GENERATION AND DIMENSIONS

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ABSTRACT. Let A be a virtually Gorenstein algebra of finite CM-type. We establish a duality between the subcategory of compact objects in the homotopy category of Gorenstein projective left A -modules and the bounded Gorenstein derived category of finitely generated right A -modules. Let R be a two-sided noetherian ring such that the subcategory of Gorenstein flat modules $R\text{-}\mathcal{GF}$ is closed under direct products. We show that the inclusion $K(R\text{-}\mathcal{GF}) \rightarrow K(R\text{-Mod})$ of homotopy categories admits a right adjoint. We introduce the notion of Gorenstein representation dimension for an algebra of finite CM-type, and establish relations among the dimension of its relative Auslander algebra, Gorenstein representation dimension, the dimension of the bounded Gorenstein derived category, and the dimension of the bounded homotopy category of its Gorenstein projective modules.

Key words: Gorenstein projective modules; Gorenstein flat modules; compactly generated homotopy categories; Gorenstein representation dimension.

1. INTRODUCTION

Gorenstein projective modules and algebras of finite CM-type are of interest in the representation theory of algebras, Gorenstein homological algebra, and in the theory of singularity categories (see e.g. [AM], [EJ], [CFH], [Buc], [Be]).

Triangulated categories, especially, derived categories, introduced by Grothendieck and Verdier([Ver]), have been playing an increasingly important role in various areas of mathematics, including representation theory, algebraic geometry, and mathematical physics. In the last decade, some of the progress has been on Brown representability in homotopy theory (see e.g. [CN], [BS2]). There are several theorems telling us about the existence and uniqueness of model structures for large classes of triangulated categories ([LO], [S]). To be able to use Brown representability Theorem, a triangulated category needs a basic property of compact generation ([N2]). Later, Krause ([K1]) and Jørgensen ([J]) have established relations between the compact generation of a triangulated category and the existence of dualizing complexes, respectively.

A major topic of current interest is the compact generation of the homotopy category of projective modules, $K(R\text{-}\mathcal{P})$, of a ring R . Jørgensen ([J]) has shown for any reasonably nice ring, $K(R\text{-}\mathcal{P})$ is compactly generated, also he has established a duality between its subcategory of compact objects and the bounded derived category of finitely presented

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right modules. This was generalized by Neeman in [N4] to arbitrary rings. He has proved that $K(R\text{-}\mathcal{P})$ is always \aleph_1 -compactly generated for any ring R .

These results raise questions on the homotopy category of Gorenstein projective modules:

(1) When is the homotopy category of Gorenstein projective modules compactly generated? What is the subcategory of compact objects? How is it related to the Gorenstein derived category of the corresponding ring introduced by Gao and Zhang ([GZ])?

(2) In this case, what is the relation between the subcategory of compact objects in the homotopy category of Gorenstein projective modules and the subcategory of compact objects in the homotopy category of projective modules?

We completely answer the question (1) and (2) for virtually Gorenstein algebras of finite CM-type (Theorem 3.2 and 3.7).

It is known that projective modules and flat modules are same over artin algebras. However, it is not true in general. For an arbitrary ring R , Neeman ([N4]) has shown the inclusion of the homotopy category of flat R -modules, $K(R\text{-}\mathcal{F})$, into $K(R\text{-Mod})$ has a right adjoint. In [K2] a criterion has been given for the existence of right approximations in cocomplete additive categories which is used to construct adjoint functors in homotopy categories.

These results raise questions on the homotopy category of Gorenstein flat modules:

(3) Can we establish a pair of adjoint functors between the homotopy category of whose Gorenstein flat modules and the homotopy category of a ring?

We completely answer the question (3) for two-sided noetherian rings such that the subcategories of Gorenstein flat modules are closed under direct products (Theorem 3.9).

The concept of dimension of a triangulated category has been introduced by Rouquier ([Ro]). He defined and studied the dimension for a triangulated category in order to prove the representation dimension of an algebra can be arbitrarily large. He also has shown for an algebra the relations among the global dimension, Auslander's representation dimension and the dimension of the bounded derived category. See [KK] and [O] for more information on this topics.

These results raise questions on the bounded Gorenstein derived category and the homotopy category of Gorenstein projective modules:

(4) What is the analogue of Auslander's representation dimension in Gorenstein homological algebra? Can we establish relations among it, the dimension of the bounded Gorenstein derived category, the dimension of the bounded homotopy category of its Gorenstein projective modules, and also the dimension of its relative Auslander algebra for an algebra of finite CM-type?

We provide such analogues and relate them by a chain of inequalities (Definition 4.2, Theorem 4.5 and Theorem 4.6).

Let us end this introduction by mentioning that model structures for Gorenstein derived categories and homotopy categories of Gorenstein projective modules are investigated, which will appear in the coming paper.

2. Preliminaries

In this section we fix notation and recall the main concepts to be used.

Let A be an artin algebra. Denote by $A\text{-Mod}$ (resp. $A\text{-mod}$) the category of (resp. finitely generated) left A -modules, and $A\text{-}\mathcal{P}$ (resp. $A\text{-proj}$) the full subcategory of (resp. finitely generated) projective A -modules. An A -module M is said to be Gorenstein projective in $A\text{-Mod}$ (resp. $A\text{-mod}$), if there is an exact sequence $P^\bullet = \cdots \rightarrow P^{-1} \rightarrow P^0 \xrightarrow{d^0} P^1 \rightarrow P^2 \rightarrow \cdots$ in $A\text{-}\mathcal{P}$ (resp. $A\text{-proj}$) with $\text{Hom}_A(P^\bullet, Q)$ exact for any A -module Q in $A\text{-}\mathcal{P}$ (resp. $A\text{-proj}$), such that $M \cong \ker d^0$ (see [EJ]). Denote by $A\text{-}\mathcal{GP}$ (resp. $A\text{-}\mathcal{Gproj}$) the full subcategory of Gorenstein projective modules in $A\text{-Mod}$ (resp. $A\text{-mod}$). For the notion of Gorenstein injective module we refer to [EJ]. We denote the subcategory of Gorenstein injective A -modules by $A\text{-}\mathcal{GI}$.

Now recall the notion of Gorenstein flat module. Denote by $A\text{-}\mathcal{F}$ the full subcategory of flat A -modules. An A -module M is said to be Gorenstein flat in $A\text{-Mod}$, if there is an exact sequence $F^\bullet = \cdots \rightarrow F^{-1} \rightarrow F^0 \xrightarrow{d^0} F^1 \rightarrow F^2 \rightarrow \cdots$ in $R\text{-}\mathcal{F}$ with $I \otimes_R F^\bullet$ exact for any injective right A -module I , such that $M \cong \ker d^0$ (see [EJ]). Denote by $A\text{-}\mathcal{GF}$ the full subcategory of Gorenstein flat modules in $A\text{-Mod}$.

A proper $A\text{-}\mathcal{Gproj}$ -resolution of A -module M in $A\text{-mod}$ is an exact sequence $E^\bullet = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ such that all $G_i \in A\text{-}\mathcal{Gproj}$, and that $\text{Hom}_A(G, E^\bullet)$ stays exact for each $G \in A\text{-}\mathcal{Gproj}$. The second requirement guarantees the uniqueness of such a resolution in the homotopy category (the Comparison Theorem; see [EJ], p.169). The coproper $A\text{-}\mathcal{Gproj}$ -resolution is defined dually.

The Gorenstein projective dimension $\mathcal{Gpdim} M$ of M in $A\text{-mod}$ is defined to be the smallest integer $n \geq 0$ such that there is an exact sequence $0 \rightarrow G_n \rightarrow \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ with all $G_i \in A\text{-}\mathcal{Gproj}$, if it exists; and $\mathcal{Gpdim} M = \infty$ if there is no such exact sequence of finite length.

A complex C^\bullet of (finitely generated) A -modules is $A\text{-}\mathcal{GP}$ (resp. $A\text{-}\mathcal{Gproj}$)-exact, if $\text{Hom}_A(G, C^\bullet)$ is exact for any $G \in A\text{-}\mathcal{GP}$ (resp. $A\text{-}\mathcal{Gproj}$). It is also called proper exact for example in [AM]. A chain map $f^\bullet : X^\bullet \rightarrow Y^\bullet$ is an $A\text{-}\mathcal{Gproj}$ -quasi-isomorphism, if $\text{Hom}_A(G, f^\bullet)$ is a quasi-isomorphism for any $G \in A\text{-}\mathcal{Gproj}$, i.e., there are isomorphisms of abelian groups $H^n \text{Hom}_A(G, f^\bullet) : H^n \text{Hom}_A(G, X^\bullet) \cong H^n \text{Hom}_A(G, Y^\bullet)$, $\forall n \in \mathbb{Z}$, $\forall G \in A\text{-}\mathcal{Gproj}$.

Following [GZ2], the (bounded) Gorenstein derived category $D_{gp}(A\text{-Mod})$ (resp. $D_{gp}^b(A)$) of A is defined as the Verdier quotient of the (bounded) homotopy category $K(A\text{-Mod})$ (resp. $K^b(A\text{-mod})$) with respect to the triangulated subcategory $K_{gpac}(A\text{-Mod})$ (resp. $K_{gpac}^b(A\text{-mod})$) of $A\text{-}\mathcal{GP}$ (resp. $A\text{-}\mathcal{Gproj}$)-acyclic complexes.

Recall from [BH, Be] an artin algebra A is of finite CM-type if there are only finitely many isomorphism classes of finitely generated indecomposable Gorenstein projective A -modules. Suppose A is an artin algebra of finite CM-type, and G_1, \dots, G_n are all the pairwise non-isomorphic indecomposable finitely generated Gorenstein projective A -modules, and $G = \bigoplus_{1 \leq i \leq n} G_i$. Set $\mathcal{Gp}(A) := \text{End}_A(G)^{\text{op}}$, which we call the relative Auslander algebra of A . It is clear that G is an $A\text{-}\mathcal{Gp}(A)$ -bimodule and $\mathcal{Gp}(A)$ is an artin algebra ([ARS, p.27]). Denote by $\mathcal{Gp}(A)\text{-mod}$ the category of finitely generated left $\mathcal{Gp}(A)$ -modules. Recall from [BR] that an artin algebra A is called virtually Gorenstein if $A\text{-}\mathcal{GP}^\perp = {}^\perp A\text{-}\mathcal{GI}$.

Let \mathcal{T} be a triangulated category with arbitrary small coproducts. Recall from [N3] that an objects $T \in \mathcal{T}$ is compact if the functor $\text{Hom}_{\mathcal{T}}(T, -)$ preserves coproducts. The full subcategory of all compact objects in \mathcal{T} will be denoted \mathcal{T}^c .

3. THE SUBCATEGORY $K(A\text{-}\mathcal{GP})^c$ OF COMPACT OBJECTS

In this section we show that if A is a virtually Gorenstein artin algebra of finite CM-type, then the subcategory of compact objects, $K(A\text{-}\mathcal{GP})^c$, of the homotopy category $K(A\text{-}\mathcal{GP})$ is triangular equivalent to the opposite category of the bounded Gorenstein derived category $D_{gp}^b(A^{\text{op}})$ of A^{op} . We also prove that if R is a two-sided noetherian ring such that the category of Gorenstein flat R -modules, $R\text{-}\mathcal{GF}$, is closed under direct products, then the inclusion of the homotopy category of Gorenstein flat R -modules, $K(R\text{-}\mathcal{GF})$, into the homotopy category $K(R\text{-}\text{Mod})$ admits a right adjoint.

Let A be a virtually Gorenstein artin algebra of finite CM-type. Then A^{op} is also a virtually Gorenstein artin algebra of finite CM-type. By [Be, Proposition 4.18] we have that $A\text{-}\mathcal{GP} = \text{Add}(A\text{-}\mathcal{Gproj})$ and $A^{\text{op}}\text{-}\mathcal{Gproj}$ is contravariantly finite in $A^{\text{op}}\text{-mod}$. Denote by $(\)^*$ the functor $\text{Hom}_A(-, A)$ which dualizes with respect to A .

Construction 3.1. *Let M be a finitely generated A -module. Then there is a proper $A^{\text{op}}\text{-}\mathcal{Gproj}$ -resolution G^\bullet with G^\bullet in $K^{-,gp^b}(A^{\text{op}}\text{-}\mathcal{Gproj})$ for M^* (See [EJ], also [GZ2]). This deduces from [HJ2, Theorem 1.6] that $G^{\bullet*}$ is a coproper $A\text{-}\mathcal{Gproj}$ -resolution of M . Set*

$$\mathcal{G}_2 := \{G^{\bullet*}[n] \mid M \in A\text{-mod}, n \in \mathbb{Z}\}$$

Theorem 3.2. *$K(A\text{-}\mathcal{GP})$ is a compactly generated triangulated category with \mathcal{G}_2 as a set of compact generators. Moreover, there is a triangle-equivalence*

$$K(A\text{-}\mathcal{GP})^c \cong D_{gp}^b(A^{\text{op}})^{\text{op}}$$

Proof. By [HJ1, Theorem 3.1] we get that \mathcal{G}_2 is a set of compact objects in $K(A\text{-}\mathcal{GP})$. Also by the proof of [G2, Theorem 2.2] we get that \mathcal{G}_2 generates $K(A\text{-}\mathcal{GP})$. This implies that $K(A\text{-}\mathcal{GP})^c$ is a full subcategory of $K(A\text{-}\mathcal{GP})$ consisting of objects which are finitely built from objects G^\bullet in \mathcal{G}_2 , using shifts, distinguished triangles, and direct summands.

Now we set $\mathcal{G}_2^* := \{G^{\bullet*} \mid G^\bullet \in \mathcal{G}_2\}$. Denote by \mathcal{D} the full subcategory of $K(A^{\text{op}}\text{-}\mathcal{GP})$ consisting of objects which are finitely built from objects in \mathcal{G}_2^* . Since the canonical chain maps $G^\bullet \rightarrow G^{\bullet**}$ and $G^{\bullet*} \rightarrow G^{\bullet***}$ are isomorphisms, it follows that

$$K(A\text{-}\mathcal{GP})^c \rightleftarrows \mathcal{D}^{\text{op}}$$

are quasi-inverse equivalences of triangulated categories.

Now we claim that \mathcal{D} consists of the objects finitely built from proper $A^{\text{op}}\text{-}\mathcal{Gproj}$ -resolutions of all finitely generated A^{op} -modules. Then by [GZ2, Theorem 3.6] we get that \mathcal{D} is triangular equivalent to $D_{gp}^b(A^{\text{op}})$. This completes the proof.

Suppose that N is a finitely generated A^{op} -module, and let

$$E^\bullet = \cdots \rightarrow E^{-2} \rightarrow E^{-1} \rightarrow E^0 \rightarrow 0$$

be a proper $A^{\text{op}}\text{-}\mathcal{Gproj}$ -resolution of N . Now

$$\widetilde{E}^\bullet = \cdots \rightarrow E^{-3} \rightarrow E^{-2} \rightarrow 0 \rightarrow \cdots$$

is a proper $A^{\text{op}}\text{-}\mathcal{G}\text{proj}$ -resolution of $Z^{-1}(\widetilde{E^\bullet})$, the (-1) st cycle module of E^\bullet . Complete $E^{0*} \rightarrow E^{-1*}$ with its cokernel,

$$E^{0*} \rightarrow E^{-1*} \rightarrow M \rightarrow 0.$$

Then $Z^{-1}(\widetilde{E^\bullet}) = M^*$. This implies $\widetilde{E^\bullet}$ is in \mathcal{G}_2^* , also E^\bullet is in \mathcal{G}_2^* . \blacksquare

Corollary 3.3. *Let A be a virtually Gorenstein algebra of finite CM-type. Then*

(1) *there exists the following recollement*

$$K(A\text{-}\mathcal{GP}) \begin{array}{c} \xleftarrow{l} \\ \xrightarrow{i} \\ \xleftarrow{r} \end{array} K(A\text{-Mod}) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} K(A\text{-Mod})/K(A\text{-}\mathcal{GP})$$

In particular, if A is Gorenstein, then we have the recollement of the form

$$K(A\text{-}\mathcal{GP}) \begin{array}{c} \xleftarrow{l} \\ \xrightarrow{i} \\ \xleftarrow{r} \end{array} K(A\text{-Mod}) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} K_{gpac}(A\text{-Mod})$$

(2) *there exists the following right recollement*

$$K_{gpac}(A\text{-Mod}) \begin{array}{c} \xleftarrow{i} \\ \xrightarrow{r} \end{array} K(A\text{-Mod}) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} D_{gp}(A\text{-Mod})$$

In particular, $D_{gp}(A\text{-Mod})$ has small Hom-sets. In this case, if A is also Gorenstein, then we have the recollement of the form

$$K_{gpac}(A\text{-Mod}) \begin{array}{c} \xleftarrow{l} \\ \xrightarrow{i} \\ \xleftarrow{r} \end{array} K(A\text{-Mod}) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} K(A\text{-}\mathcal{GP})$$

Proof. (1) By Theorem 3.2 we get that $K(A\text{-}\mathcal{GP})$ is compactly generated. Since the inclusion i naturally preserves coproducts and products, it follows from [N2, Theorem 4.1] that i admits a right adjoint r , also a left adjoint l . So by [Mi, Theorem 2.2] we have the following recollement

$$K(A\text{-}\mathcal{GP}) \begin{array}{c} \xleftarrow{l} \\ \xrightarrow{i} \\ \xleftarrow{r} \end{array} K(A\text{-Mod}) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} K(A\text{-Mod})/K(A\text{-}\mathcal{GP})$$

If A is Gorenstein, then by [G1, Theorem 2.7] we have a triangle-equivalence

$$K(A\text{-Mod})/K(A\text{-}\mathcal{GP}) \cong K_{gpac}(A\text{-Mod}).$$

This completes the proof of (1).

(2) By [Be, Theorem 4.10] we have that every module in $A\text{-}\mathcal{GP}$ is a filtered colimit of modules in $A\text{-}\mathcal{Gproj}$. This follows that $C_{gpac}(A\text{-Mod})$ is closed under filtered colimits. By minor modifications of the proof of [K2, Lemma 2(4)] we get that $C_{gpac}(A\text{-Mod})$ is closed under α -pure subobjects for some regular cardinal α . Thus by [K2, Theorem 4] we get that every complex in $K(A\text{-Mod})$ admits a right $K_{gpac}(A\text{-Mod})$ -approximation. Applying [N5, Proposition 1.4], it follows that the inclusion $i : K_{gpac}(A\text{-Mod}) \rightarrow K(A\text{-Mod})$ admits a right adjoint r . Therefore we obtain from [Mi, Theorem 2.2] the following right recollement

$$K_{gpac}(A\text{-Mod}) \begin{array}{c} \xleftarrow{i} \\ \xrightarrow{r} \end{array} K(A\text{-Mod}) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} D_{gp}(A\text{-Mod})$$

By [GZ1, Proposition I.1.3] we know that the right adjoint of the quotient functor $K(A\text{-Mod}) \rightarrow D_{gp}(A\text{-Mod})$ is fully faithful. Therefore $D_{gp}(A\text{-Mod})$ has small Hom-sets.

Since A is virtually Gorenstein of finite CM-type, it follows from the proof of (1) that the inclusion $K(A\text{-}\mathcal{GP}) \rightarrow K(A\text{-Mod})$ admits a right adjoint. Moreover, if A is Gorenstein, then by [G1, Theorem 2.7] we get that $K(A\text{-}\mathcal{GP})^\perp = K_{gpac}(A\text{-Mod})$. This means that the inclusion $K_{gpac}(A\text{-Mod}) \rightarrow K(A\text{-Mod})$ admits a left adjoint and the composition $K(A\text{-}\mathcal{GP}) \rightarrow K(A\text{-Mod}) \rightarrow D_{gp}(A\text{-Mod})$ is a triangle-equivalence. This completes the proof of (2). \blacksquare

Corollary 3.4. *Let A be a Gorenstein algebra of finite CM-type. Then the canonical functor $D_{gp}(A\text{-Mod}) \rightarrow D(A\text{-Mod})$ admits left and right adjoints that are fully faithful. The left adjoint preserves compactness and its restriction to compact objects identifies with the inclusion $K^b(A\text{-proj}) \rightarrow K^b(A\text{-Gproj})$.*

Proof. Since A is Gorenstein of finite CM-type, it follows from Theorem 3.2 that $K(A\text{-GP})$ is compactly generated, and $K(A\text{-GP})^c \cong K^b(A\text{-Gproj})$. By Corollary 3.3(2) we have a triangle-equivalence $D_{gp}(A\text{-Mod}) \cong K(A\text{-GP})$. This implies that $D_{gp}(A\text{-Mod})^c \cong K^b(A\text{-Gproj})$. Note that the canonical functor $F : D_{gp}(A\text{-Mod}) \rightarrow D(A\text{-Mod})$ preserves set-indexed coproducts and products. Thus by [N2, Theorem 4.1] we get that F admits a left adjoint and a right adjoint, also the left adjoint preserves compactness. This follows from [N1, Theorem 2.1] that the restriction of this left adjoint to compact objects identifies with the inclusion $K^b(A\text{-proj}) \rightarrow K^b(A\text{-Gproj})$. Since F is a quotient functor, it follows from [GZ1, Proposition I.1.3] that these adjoints are fully faithful. ■

Next we will compare the subcategory of compact objects in the homotopy category of projective modules with the subcategory of compact objects in the homotopy category of Gorenstein projective modules. We first recall the construction of compact generators of the homotopy category of projective A -modules for an artin algebra A we will use.

Let A be an artin algebra and M a finitely generated A -module. Then for M^* there is a canonical quasi-isomorphism $P^\bullet \rightarrow M^*$ with P^\bullet in the homotopy category $K^{-,b}(A^{\text{op}}\text{-proj})$. We consider the collection of the form $P^{\bullet\bullet}[n](n \in \mathbb{Z})$. Denote by \mathcal{G}_1 the set of one object from each such isomorphism class.

Lemma 3.5. *([J, Theorem 2.4 and 3.2]) The homotopy category $K(A\text{-P})$ is a compactly generated triangulated category with \mathcal{G}_1 as a set of compact generators. Moreover, there is a triangle-equivalence*

$$K(A\text{-P})^c \cong D^b(A^{\text{op}})^{\text{op}}$$

Theorem 3.6. *Let A be a virtually Gorenstein algebra of finite CM-type. Then we have*

- (1) $K(A\text{-GP})/K(A\text{-P})$ is a compactly generated triangulated category.
- (2) the localisation sequence of triangulated categories

$$K(A\text{-P}) \xrightarrow{i} K(A\text{-GP}) \xrightarrow{q} K(A\text{-GP})/K(A\text{-P})$$

yields, by restriction to compact objects, a sequence of functors

$$D^b(A^{\text{op}})^{\text{op}} \rightarrow D_{gp}^b(A^{\text{op}})^{\text{op}} \rightarrow (K(A\text{-GP})/K(A\text{-P}))^c.$$

Moreover the induced functor

$$F : D_{gp}^b(A^{\text{op}})^{\text{op}}/D^b(A^{\text{op}})^{\text{op}} \rightarrow (K(A\text{-GP})/K(A\text{-P}))^c$$

is fully faithful, and identifies $D_{gp}^b(A^{\text{op}})^{\text{op}}/D^b(A^{\text{op}})^{\text{op}}$ with a subcategory of $(K(A\text{-GP})/K(A\text{-P}))^c$ whose épaisse closure is all of $(K(A\text{-GP})/K(A\text{-P}))^c$.

Proof. [G2, Theorem 2.6] implies (1). Now we prove (2). By Lemma 3.5 we know that $K(A\text{-P})$ is a compactly generated triangulated category with \mathcal{G}_1 as a set of compact generators, and there is an equivalence of triangulated categories $K(A\text{-P})^c \cong D^b(A^{\text{op}})^{\text{op}}$. By Theorem 3.2 we get that $K(A\text{-GP})$ is a compactly generated triangulated category with \mathcal{G}_2 as a set of compact generators, and there is an equivalence of triangulated categories $K(A\text{-GP})^c \cong D_{gp}^b(A^{\text{op}})^{\text{op}}$.

Let $\{G_i^\bullet\}_{i \in I}$ be any family objects in $K(A\text{-GP})$. Then $\text{Hom}_{K(A\text{-GP})}(iP, \coprod_{i \in I} G_i^\bullet) = \text{Hom}_{K(A\text{-GP})}(P, \coprod_{i \in I} G_i^\bullet) \cong \coprod_{i \in I} \text{Hom}_{K(A\text{-GP})}(P, G_i^\bullet) = \coprod_{i \in I} \text{Hom}_{K(A\text{-GP})}(iP, G_i^\bullet)$ for

each module $P \in A\text{-proj}$. So by [CFH, Proposition 2.6] the inclusion $i : K(A\text{-}\mathcal{P}) \hookrightarrow K(A\text{-}\mathcal{GP})$ preserves compact objects.

Applying [N1, Theorem 2.1] to the homotopy category $K(A\text{-}\mathcal{GP})$ and $K(A\text{-}\mathcal{P})$, we get that i carries $K(A\text{-}\mathcal{P})^c$ to $K(A\text{-}\mathcal{GP})^c$, q carries $K(A\text{-}\mathcal{GP})^c$ to $(K(A\text{-}\mathcal{GP})/K(A\text{-}\mathcal{P}))^c$, the natural functor $\tilde{F} : K(A\text{-}\mathcal{GP})^c/K(A\text{-}\mathcal{P})^c \rightarrow (K(A\text{-}\mathcal{GP})/K(A\text{-}\mathcal{P}))^c$ is fully faithful, and any object in $K(A\text{-}\mathcal{GP})^c/K(A\text{-}\mathcal{P})^c$ is a direct summand of some object in $(K(A\text{-}\mathcal{GP})/K(A\text{-}\mathcal{P}))^c$. This completes the proof of (2). \blacksquare

Recall that an additive category \mathcal{C} is idempotent-complete if every idempotent morphism splits. Any additive category admits an idempotent completion $l : \mathcal{C} \rightarrow \mathcal{C}^\natural$. Moreover, if \mathcal{C} is triangulated, then \mathcal{C}^\natural inherits a unique structure of triangulated category such that l is a triangle functor (see [BS]).

Remark 3.7. *Let A be a virtually Gorenstein algebra of finite CM-type. Theorem 3.6 implies that the idempotent completion of $D_{gp}^b(A^{\text{op}})^{\text{op}}/D^b(A^{\text{op}})^{\text{op}}$ and $(K(A\text{-}\mathcal{GP})/K(A\text{-}\mathcal{P}))^c$ are triangulated equivalent, also $D^b(A)$ can be viewed as a triangulated subcategory of $D_{gp}^b(A)$. But in general, this doesn't hold.*

In general, a Gorenstein flat module is not necessarily Gorenstein projective for a ring. Next we will establish a pair of adjoint functors between the homotopy category of its Gorenstein flat modules and the homotopy category of a nice ring.

Theorem 3.8. *Let R be a two-sided noetherian ring such that $R\text{-}\mathcal{GF}$ is closed under direct products. Then the inclusion $K(R\text{-}\mathcal{GF}) \rightarrow K(R\text{-}\text{Mod})$ has a right adjoint.*

Proof. By [EEI, Theorem 4.3] we get that every complex X^\bullet in $K(R\text{-}\text{Mod})$ admits a $K(R\text{-}\mathcal{GF})$ -precover. Hence by [N5, Proposition 1.4] we get that the inclusion $K(R\text{-}\mathcal{GF}) \rightarrow K(R\text{-}\text{Mod})$ has a right adjoint. \blacksquare

Now we show an interesting phenomenon. Let M be any R -module, and consider the complex below

$$\cdots \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow \cdots$$

The existence of a right adjoint to the inclusion gives us a morphism $Z^\bullet \rightarrow M$, in the category $K(R\text{-}\text{mod})$,

$$\begin{array}{ccccccc} \cdots & \longrightarrow & Z^{-1} & \longrightarrow & Z^0 & \longrightarrow & Z^1 \longrightarrow \cdots \\ & & \downarrow & & \downarrow \rho & & \downarrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 \longrightarrow \cdots \end{array}$$

where the complex Z^\bullet is a complex of Gorenstein flat R -modules. Furthermore, given any map $\varphi : F^\bullet \rightarrow M$, with F a Gorenstein flat R -module, we have a factorization of φ :

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & F & \longrightarrow & 0 \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & Z^{-1} & \longrightarrow & Z^0 & \longrightarrow & Z^1 \longrightarrow \cdots \\ & & \downarrow & & \downarrow \rho & & \downarrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 \longrightarrow \cdots \end{array}$$

This shows the map $\rho : Z^0 \rightarrow M$ is a Gorenstein flat precover for M .

Corollary 3.9. *Let R be a two-sided noetherian ring such that $R\text{-}\mathcal{GF}$ is closed under direct products. Then the inclusion $i : K(R\text{-}\mathcal{F}) \rightarrow K(R\text{-}\mathcal{GF})$ has a right adjoint.*

Proof. By [N5, Theorem 3.2] we get that the inclusion $i_1 : K(R\text{-}\mathcal{F}) \rightarrow K(R\text{-Mod})$ has a right adjoint j_1 . By Theorem 3.8 we get the inclusion $i_2 : K(R\text{-}\mathcal{GF}) \rightarrow K(R\text{-Mod})$ has a right adjoint j_2 . Since we have a series of isomorphisms for any $F^\bullet \in K(R\text{-}\mathcal{F})$ and $Z^\bullet \in K(R\text{-}\mathcal{GF})$

$$\begin{aligned} \text{Hom}_{K(R\text{-}\mathcal{GF})}(iF^\bullet, Z^\bullet) &\cong \text{Hom}_{K(R\text{-Mod})}(i_2 i F^\bullet, i_2 Z^\bullet) \\ &= \text{Hom}_{K(R\text{-Mod})}(i_1 F^\bullet, i_2 Z^\bullet) \\ &\cong \text{Hom}_{K(R\text{-}\mathcal{F})}(F^\bullet, j_1 i_2 Z^\bullet), \end{aligned}$$

it follows that $i : K(R\text{-}\mathcal{F}) \rightarrow K(R\text{-}\mathcal{GF})$ admits a right adjoint $j_1 i_2$. \blacksquare

Enochs and Estrada in [EE] defined the category of quasi-coherent \mathfrak{R} -modules, $\mathfrak{R}\text{-mod}$, where \mathfrak{R} is a representation by rings of a quiver Q . They aim to understand the category of quasi-coherent sheaves $\mathcal{O}coX$ on a scheme X via the equivalence between it and $\mathfrak{R}\text{-mod}$ for some quiver Q and ring \mathfrak{R} . They also found that if X is a locally Gorenstein scheme, then $\mathcal{O}coX$ is a Gorenstein category. Now, an example of above theorem arise.

Example 3.10. *Let A be a commutative noetherian ring and $(X, \mathcal{O}_X) \subseteq \mathbb{P}^n(A)$ be a locally Gorenstein scheme. Denote by $\mathcal{Q}coX$ the category of quasi-coherent sheaves on X and by $\mathcal{GF}X$ the subcategory of Gorenstein flat quasi-coherent \mathcal{O}_X -modules. Then the inclusion $K(\mathcal{GF}X) \rightarrow K(\mathcal{Q}coX)$ has a right adjoint.*

Proof. Since X is a locally Gorenstein scheme, following the notations in Section 3 in [EEG-R], $\mathcal{Q}coX$ is equivalent to the category $\mathfrak{R}\text{-mod}$ as abelian categories, where \mathfrak{R} is an associated ring such that $\mathfrak{R}(v)$ is a commutative Gorenstein ring for any vertex v .

By [EX, Lemma 3.5] we know that $\mathfrak{R}(v)$ -module N is Gorenstein flat if and only if $N^+ = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$ is a Gorenstein injective $\mathfrak{R}(v)$ -module for all vertex v . So by [EEG-R, Corollary 3.13] we get that M is a Gorenstein flat \mathfrak{R} -module if and only if $M(v)$ is a Gorenstein flat $\mathfrak{R}(v)$ -module for all vertex v . Since $\mathfrak{R}(v)$ has a dualizing complex for all vertex v , it follows from [CFH, Theorem 5.7] that $\mathfrak{R}(v)\text{-}\mathcal{GF}$ is closed under direct products. This deduces that $\mathfrak{R}\text{-}\mathcal{GF}$ is closed under direct products. By Theorem 3.8 we get the inclusion $K(\mathfrak{R}\text{-}\mathcal{GF}) \rightarrow K(\mathfrak{R}\text{-Mod})$ admits a right adjoint. This deduces that $K(\mathcal{GF}X) \rightarrow K(\mathcal{Q}coX)$ has a right adjoint. \blacksquare

4. Gorenstein representation Dimension

In this section we introduce the notion of Gorenstein representation dimension for an algebra of finite CM-type, and establish relations among the dimension of its relative Auslander algebra, Gorenstein representation dimension, the dimension of the bounded Gorenstein derived category, and also the dimension of the bounded homotopy category of Gorenstein projective modules.

Let \mathcal{T} be a triangulated category, and $M \in \mathcal{T}$. We set

$$\langle M \rangle = \langle M \rangle_1 = \text{add}\{M[i] \mid i \in \mathbb{Z}\}$$

$$\langle M \rangle_{n+1} = \text{add}\{X \mid \exists M' \rightarrow X \rightarrow M'' \rightarrow M'[1] \text{ with } M' \in \langle M \rangle, M'' \in \langle M \rangle_n\}$$

Recall from [Ro] that the dimension of a triangulated category \mathcal{T} is the number

$$\dim \mathcal{T} = \inf\{n \in \mathbb{N} \mid \text{there exists a } M \in \mathcal{T} \text{ with } \langle M \rangle_{n+1} = \mathcal{T}\}$$

For a subcategory $\mathcal{C} \subseteq \mathcal{T}$ the dimension is defined to be $\dim_{\mathcal{T}} \mathcal{C} = \inf\{n \mid \exists M \in \mathcal{T} : \mathcal{C} \subseteq \langle M \rangle_{n+1}\}$. We first have

Lemma 4.1. *Let A be a finite dimensional k -algebra of finite CM-type over a field k and $\mathcal{G}\text{pdim} X = n$ for some $X \in A\text{-mod}$. Then $X \notin \langle G \rangle_n$.*

Proof. The proper $A\text{-}\mathcal{G}\text{proj}$ -resolution

$$\Omega_G^n X \rightarrow G_{n-1} \rightarrow G_{n-2} \rightarrow \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow X$$

gives rise to a sequence of maps

$$X \rightarrow \Omega_G X[1] \rightarrow \cdots \rightarrow \Omega_G^{n-1} X[n-1] \rightarrow \Omega_G^n X[n]$$

in $D_{gp}^b(A)$. They are all $\langle G \rangle$ -ghosts, and their composition is non-zero. Hence the claim follows from the ghost lemma in the sense of Rouquier [Ro]. \blacksquare

So we try to introduce the notion of Gorenstein representation dimension. Of particular interest to us are $\dim D_{gp}^b(A)$ and $\dim_{D_{gp}^b(A)}(A\text{-mod})$.

Definition 4.2. *Let A be an artin algebra of finite CM-type. The Gorenstein representation dimension of A is defined as*

$$\text{Grepdim} A = \min\{M\text{-resol.dim}(A\text{-mod}) \mid M \in A\text{-mod such that}$$

$$G \oplus \nu(G) \in \text{add} M\} + 2.$$

An A -module M realizing the minimum above is called the Gorenstein Auslander generator.

Remark 4.3. *Let A be an artin algebra of finite CM-type. Then $\text{repdim} A \leq \text{Grepdim} A$ by the definition of representation dimension (see [Au]). Also, if A is CM-free (i.e. $A\text{-}\mathcal{G}\mathcal{P} = A\text{-}\mathcal{P}$), then these two definitions coincide.*

Lemma 4.4. *Let A be an artin algebra of finite CM-type. Then*

$$\text{Grepdim} A = \min\{\text{gl.dim} \text{End}_A(M) \mid M \in A\text{-mod such that } G \oplus \nu(G) \in \text{add} M\}$$

Proof. For any $M \in A\text{-mod}$ with $G \oplus \nu(G) \in \text{add} M$, we easily see that M is a generator and cogenerator. Hence the claim follows from [Au]. \blacksquare

Now we will establish relations among above-mentioned dimensions for an algebra of finite CM-type.

Theorem 4.5. *Let A be a finite dimensional k -algebra of finite CM-type. Let $M \in A\text{-mod}$ be a Gorenstein Auslander generator, and $X \in A\text{-mod}$. Then for any $n \in \mathbb{Z}$ we have*

$$M\text{-resol.dim} X \leq n \implies X \in \langle M \rangle_{n+1}.$$

In particular,

$$\text{Grepdim} A \geq \dim_{D_{gp}^b(A)}(A\text{-mod}) + 2.$$

Proof. This follows immediately from the fact that $A\text{-}\mathcal{G}\text{proj}$ -acyclic short exact sequences in $A\text{-mod}$ turn into distinguished triangles in $D_{gp}^b(A)$. \blacksquare

Theorem 4.6. *Let A be a finite dimensional k -algebra of finite CM-type. Then*

$$\text{Grepdim} A \geq \dim D_{gp}^b(A) \geq \dim \mathcal{G}\mathcal{P}(A) - 1$$

Proof. Let M be a Gorenstein Auslander generator. By induction we get that every bounded complex of A -mod is A - $\mathcal{G}proj$ -quasi-isomorphic to a bounded complex of $\text{add}M$, i.e., the canonical functor $K^b(\text{add}M) \rightarrow D_{gp}^b(A)$ is essentially surjective. Notice that we have canonical equivalences $K^b(\text{add}M) \cong K^b(\text{End}_A(M)\text{-proj}) \cong D^b(\text{End}_A(M))$ and that $\dim D^b(\text{End}_A(M)) \leq \text{gl.dim End}_A(M)$. These deduce that $\text{Grepdim} A \geq \dim D_{gp}^b(A)$.

Since A is of finite CM-type, we have that $D_{gp}^b(A)$ is triangular equivalent to $K^{-,gp^b}(A\text{-}\mathcal{G}proj)$, and so G is a generator of $D_{gp}^b(A)$. Note from [BS1, Theorem 2.8] that $D_{gp}^b(A)$ is idempotent split, and that $\text{Hom}_{D_{gp}^b(A)}(G, -) : D_{gp}^b(A) \rightarrow \text{Ab}$ is a cohomological functor such that $\text{Hom}_{D_{gp}^b(A)}^*(G, G) = \mathcal{G}p(A)$ is an artin k -algebra. Hence by [BIKO, Theorem 4.5] we get that $\dim D_{gp}^b(A) \geq \dim \mathcal{G}p(A) - 1$. This completes the proof. ■

Theorem 4.7. *Let A be a virtually Gorenstein algebra of finite CM-type. Then we have $\dim K(A\text{-}\mathcal{GP}) \geq \dim \mathcal{G}p(A) - 1$.*

Proof. Since A is virtually Gorenstein of finite CM-type, it follows from [Be, Theorem 4.10] that G is a generator of $K(A\text{-}\mathcal{GP})$. Note that $K(A\text{-}\mathcal{GP})$ is idempotent split and $\text{Hom}_{K(A\text{-}\mathcal{GP})}(G, -) : K(A\text{-}\mathcal{GP}) \rightarrow \text{Ab}$ is a cohomological functor such that $\text{Hom}_{K(A\text{-}\mathcal{GP})}^*(G, G) = \mathcal{G}p(A)$ is an artin k -algebra. Hence by [BIKO, Theorem 4.5] we get that $\dim K(A\text{-}\mathcal{GP}) \geq \dim \mathcal{G}p(A) - 1$. ■

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